

Probabilistic Calibration of Predictive Distributions

Calibrazione di densità previsive

Roberto Casarin and Tillman Gneiting and Francesco Ravazzolo

Abstract This contribution studies the calibration approach recently proposed in [2] for predictive densities. We consider a set of simulated experiments in order to study the effectiveness of the method.

Abstract *Questo contributo studia un approccio alla calibrazione delle densità previsive recentemente proposto in [2]. Viene considerata una serie di esperimenti in simulazione al fine di verificare l'efficienza del metodo proposto.*

Key words: Density calibration, Predictive distributions.

1 Introduction

The use of predictive densities in providing forecast on some economic quantities of interest received a lot of attention in the recent years. An increasing stream of literature focused on the optimal combination of predictive densities (e.g., see [9], [6], [1], and [5]). Another important issue in density forecast relates to the calibration property of the density (e.g., see [3] and [4]). The properties of the predictive densities and calibration procedures have been recently investigated in [7] and [8]. [2] extended the beta calibration approach of [8]. They proposed a general calibration and combination approach that estimates calibration parameters and combination weights in different and possibly overlapping regions of the support set of the calibrated predictive density. More specifically they suggested the use of mixtures of

Roberto Casarin
University Ca' Foscari of Venice e-mail: r.casarin@unive.it

Tillman Gneiting
Heidelberg Institute for Theoretical Studies and Karlsruhe Institute of Technology e-mail:
Tilmann.Gneiting@h-its.org

Francesco Ravazzolo
Norges Bank and BI Norwegian Business School e-mail: Francesco.Ravazzolo@norges-bank.no

beta densities to obtain a better calibration of the predictive density; and also showed that the mixture can be written in terms of continuous composition of local optimal linear combination schemes, with weights which depend on the variable of interest. In this contribution we assume a linear combination scheme with equal weights and focus on the calibration part of the approach proposed in [2]. Section 2 presents the calibration model and section 3 illustrates the results from the simulation experiments.

2 A calibration model

Let $F_{jt}(y_t)$, $j = 1, \dots, M$, be a set of predictive cumulative density functions (cdf) from different models available at time $t - 1$, for a variable of interest y_t , and conditional on the information set, $\mathcal{F}_{t-1}$, available at time $t - 1$. We assume the variable of interest is continuous, and that each predictive distribution admits a probability density function (pdf) $f_{jt}(y_t)$.

Following [7] and [8], a calibrated predictive cdf can be defined by a continuous transform of the combined density. In this contribution we assume a linear combination model with equal weights, that is $\omega_j = 1/M$ for $j = 1, \dots, M$. The combined cdf is

$$H(y_t) = \sum_{i=1}^M \omega_i F_{it}(y_t) \quad (1)$$

with pdf

$$h(y_t) = \sum_{i=1}^M \omega_i f_{it}(y_t) \quad (2)$$

The calibration scheme proposed in [2] is

$$F_t(y_t) = g(H(y_t)), \quad (3)$$

where $g : [0, 1] \mapsto [0, 1]$ is the following calibration function

$$f_t(y_t) = \sum_{k=1}^2 w_k f_{\alpha_k, \beta_k}(H(y_t)) h(y_t), \quad (4)$$

where $f_{\alpha, \beta}(x)$ the pdf of a beta distribution, $\mathcal{B}e(\alpha, \beta)$, evaluated at x , w_1 and $w_2 = 1 - w_1$ are mixture probabilities and (α_1, α_2) and (β_1, β_2) are the calibration parameters.

The likelihood function for this calibration model is

$$L(Y|\theta) = \prod_{t=1}^T \left(\sum_{k=1}^2 w_k f_{\alpha_k, \beta_k}(H(y_t)) h(y_t) \right) \quad (5)$$

with $\theta = (w_1, w_2, \alpha_1, \alpha_2, \beta_1, \beta_2)$. In the linear pooling case, the log-score rule associated to this likelihood is not a weighted logarithmic scoring rule, which is an improper rule, and is instead a weighted sum of densities. We follow here a Bayesian approach and apply the estimation algorithm presented in [2].

3 A numerical study

In the simulation study we assume that a combined predictive distribution can be obtained from the two normal predictive distributions with different location and scale parameters, $\mathcal{N}(-1, 1)$ and $\mathcal{N}(2, 2)$, where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with location μ and scale σ . Moreover, we assume that the data are generated by the following mixture of the three normal distributions

$$y_t \stackrel{i.i.d.}{\sim} \frac{1}{6}\mathcal{N}(-1, 0.4) + \frac{2}{6}\mathcal{N}(0, 0.4) + \frac{3}{6}\mathcal{N}(2, 0.4), \quad t = 1, \dots, 500.$$

Let us denote with $\varphi(x|\mu, \sigma^2)$ and $\Phi(x|\mu, \sigma^2)$ the pdf and cdf respectively of a $\mathcal{N}(\mu, \sigma^2)$. We compare the following alternative models

1. Ideal model (I)

$$f(y) = p_1\varphi(y|-2, 0.4) + p_2\varphi(y|2, 0.4) + p_3\varphi(y|2, 0.4),$$

where (p_1, p_2, p_3) are the mixture probabilities used in the data generating process.

2. Non-calibrated model (NC)

$$f(y|\theta) = 0.5\varphi(y|-1, 1) + 0.5\varphi(y|2, 2),$$

3. Beta calibration model (BC)

$$f(y|\theta) = f_{\alpha, \beta}(H(y))h(y),$$

where $\theta = (\alpha, \beta)$, $h(y) = 0.5\varphi(y|-1, 1) + 0.5\varphi(y|2, 2)$ and $H(y) = 0.5\Phi(y|-1, 1) + 0.5\Phi(y|2, 2)$.

4. Two-component beta mixture calibration model (BMC)

$$f(y|\theta) = (wf_{\alpha_1, \beta_1}(H(y|\omega)) + (1-w)f_{\alpha_2, \beta_2}(H(y|\omega)))h(y|\omega),$$

where $\theta = (w, \alpha_1, \alpha_2, \beta_1, \beta_2)$, and $h(y)$ and $H(y)$ have been defined as in the BC model.

The top graph of Fig. 1 shows the empirical cdfs of different sequences of probability integral transforms (PITs)

$$z_t = \int_{-\infty}^{y_t} f(y)dy, \quad t = 1, \dots, 500, \quad (6)$$

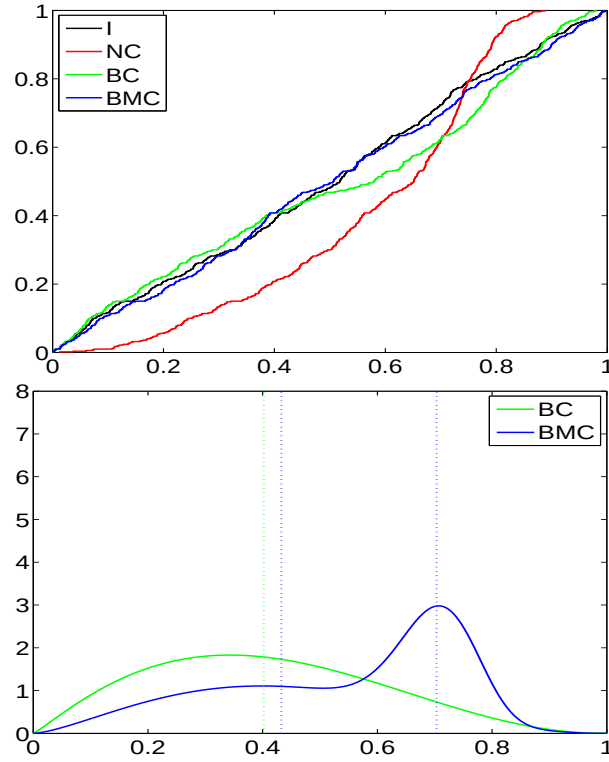


Fig. 1 Cumulative density functions of the PITs (top) for different calibration models and the Beta calibration functions for the BC and BMC models.

where $f(y)$ is the pdf of one of the models: I, NC, BC, and BMC given above. In all the experiments, the PITs of the non-calibrated model (red lines) are far from those of the ideal model (black lines). The BC does not provide well calibrated densities. The BC cdf (green line) is closer to uniformity than the NC model, but it has difficulties in calibrating the density in some parts of the support set. The two-component BMC is providing instead better calibrated PITs, since they are closer to the one of the ideal model.

The two-component beta calibration achieves a more flexible deformation of the combination cdf providing a calibrated cdf (magenta line) which is closer to the uniform cdf. The deformation obtained with the beta density and the beta mixture, in the different part of the support set are given in the bottom graph of Fig. 1.

References

1. Billio, M., Casarin, R., Ravazzolo, F., Van Dijk, H.: Time-varying Combinations of Predictive Densities using Nonlinear Filtering. *Journal of Econometrics*, **177**(2), 213–232, (2013)
2. Casarin, R., Gneiting, T., Ravazzolo, F.: Bayesian Nonparametric Calibration and Combination of Predictive Distributions, Dep. of Economics WP, University Ca' Foscari of Venice (2014)
3. Dawid, A.P.: The well-calibrated Bayesian. *Journal of the American Statistical Association*, **77**, 605–610, (1982).
4. Dawid, A.P.: Calibration-based empirical probability. *The Annals of Statistics*, **13**, 1251–1273, (1985).
5. Fawcett, N., Kapetanios, G., Mitchell, J., and Price, S.: Generalised density forecast combinations. Technical report, Warwick Business School, (2013).
6. Geweke, J., Amisano, G.: Optimal prediction pools. *Journal of Econometrics*, **164**(2), 130–141, (2010)
7. Gneiting, T., Raftery, A.E.: Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, **102**, 359–378, (2007).
8. Gneiting, T., Ranjan R.: Comparing density forecasts using threshold- and quantile-weighted scoring rules. *Journal of Business and Economic Statistics*, **29**, 411–422, (2011).
9. Hall, S.G., Mitchell, J.: Combining density forecasts. *International Journal of Forecasting*, **23**, 1–13, (2007)